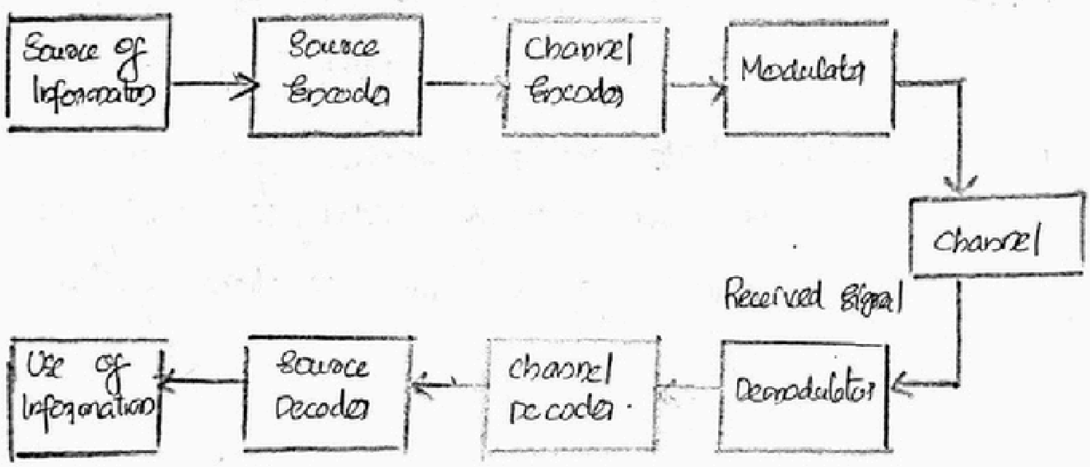
MODULE

SYLLABUS

* Overview of Random Variables and Random process:
Random variables – continuous and Discrete, random process Stationarity, Autocorrelation and power spectral density, Transmission of Random Process through LTI Systems, PSD, AWGN
* Pulse Code modulation (PCM):
Pulse Modulation, Sampling process, Performance Comparison of various sampling techniques Aliasing, Reconstruction, PAM, Quantization, Noise in PCM system.
* Modifications of PCM:
Delta modulation, ~~PD~~ DPCM, ADPCM, ADM, performance comparison of various pulse modulation schemes, Lines code, PSD of various Line codes.

ELEMENTS OF DIGITAL COMMUNICATION SYSTEMS



Source of Information.
There are 2 kinds of source of information are available such as
1. Analog Information Sources 2. Digital Information Sources.

Analog Information Sources
Microphone actuated by a speech, TV camera scanning a ~~scen~~ scene, continuous amplitude signals.

Digital Information Sources.
These are teletype or numerical output of computer which consists of a sequence of discrete symbols or letters. Analog information is transformed into discrete information through the process of sampling and

quantizing.

* **Source Encoder/Decoder**

The source encoder converts the input i.e, symbol sequence into a binary sequence of '0's and '1's by assigning code words to the symbols in the input sequence.
At the receiver, the source decoder converts the binary output of the channel decoder into a symbol sequence.

* **channel Encoder/Decoder**

Error control is accomplished by the channel coding operation that consists of systematically adding extra bits to the output of the source coder. These extra bits donot convey any information but helps the receiver to detect and/or correct some of the errors in the information bearing bits. There are 2 methods of channel coding:

1. Block Coding.

Encoder takes a block of 'k' information bits from the source encoder and adds 'r' error control bits, where 'r' is dependent on 'k' and error control desired.

2. Convoluting coding.

The information bearing message stream is encoded in a continuous fashion by continuously interleaving information bits and error control bits. The channel decoder recovers the information bearing bits from the coded binary system. Error detection and possible correction is also performed by the channel decoder. Important parameters of coder/decoder are:

- Method of coding
- Effeciency
- Error control capabilities
- Complexity of the circuit.

* **Modulator.**

It converts the input bit stream into an electrical waveform suitable for transmission over the communication channel. Modulator can be effectively used to minimize the effects of channel noise, to match frequency spectrum of transmitted signal with channel characteristics, to provide the capability to multiplex many signals.

* **Demodulator.**

The extraction of the message from the information bearing waveform produced by the modulator is accomplished by the demodulator. The output of the demodulator is bit stream. The important parameter is the method of demodulation.

* Channel.

The channel provides the electrical connection between the source and destination. The different channels are:

* Pair of wires
* Coaxial cable
* Optical fibre
* Radio channel
* Satelite channel or combination of any of these.

Advantages of Digital Communication.

- The effect of distortion, noise & interference is less in a digital communication system. This is because the disturbance must be large enough to change the pulse from one state to the other.
- Regenerative repeaters can be used at fixed distance along the link, to identify and regenerate a pulse before it is degraded to an ambiguous state
- Digital circuits are more reliable & cheaper compared to analog circuits.
- The hardware implementation is more flexible than analog hardware because of the use of microprocessors, VLSI chips etc.
- Signal processing functions like encryption, compression can be employed to maintain the secrecy of the information.
- Error detecting and Error correcting codes improve the system performance by reducing the probability of error.
- Combining digital signals using TDM is simpler than combining analog signals using FDM. The different types of signals such as data, telephone, TV can be treated as identical signals in transmission and switching in a digital communication system.
- We can avoid signal jamming using spread spectrum technique.

Disadvantages of Digital Communication.

- Large system Bandwidth:
Digital transmission requires a large system bandwidth to communicate the same information in a digital format as compared to analog format
- System Synchronization:
Digital detection requires System Synchronization whereas the analog signals generally have no such requirement.

Comparison of Analog and Digital Communication

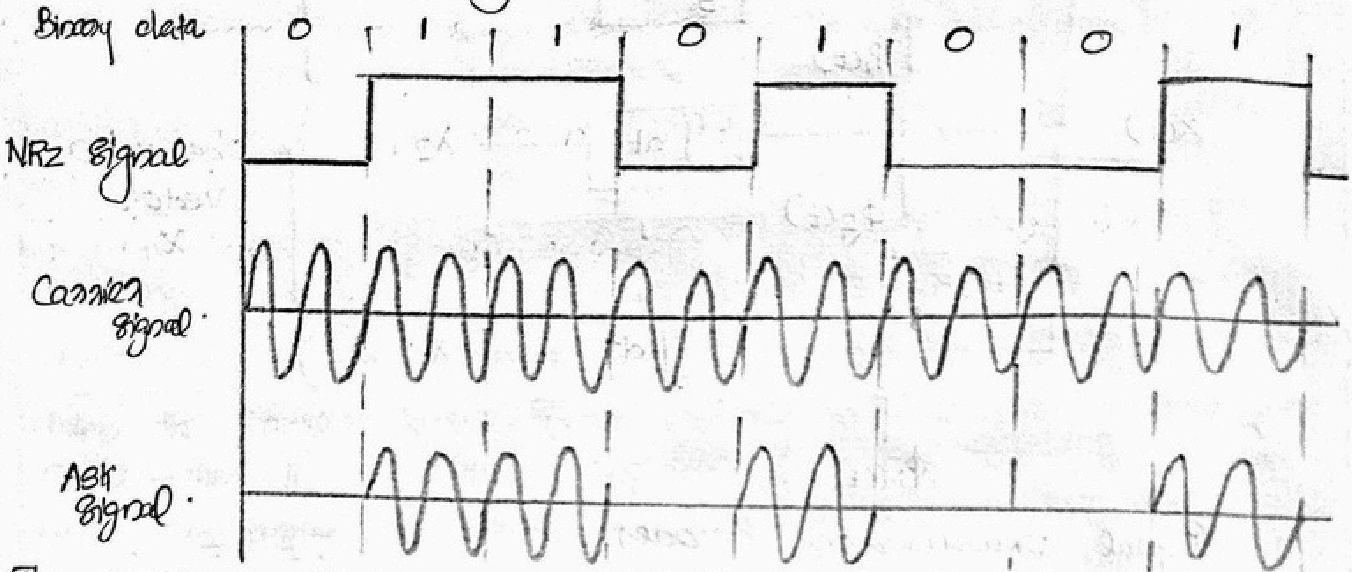
Sl No.	Analog Communication	Digital Communication
1.	The signal can take up voltage level corresponding to any real number.	The signal can take up only one among two voltage level corresponding to 1 and 0
2	Once the signal (voltage) is ~~computed~~ corrupted with noise it is difficult to recover the correct value	Even if it is corrupted by channel noise it is possible to recover the original information by using a suitable threshold at the receiver.
3	Repeaters in analog communication are amplifiers which also amplifies noise thereby degrading the quality of the system.	Repeaters in digital communication can be wave regenerators which produce new waveforms after recovering the original information from the received waveform & hence noise does not accumulate through a digital communication link.
4	Since the voltages can take infinite levels, it would need infinite bits to represent data	Since the voltage can take only finite number of values, it requires finite number of bits for propogation.
5.	Transmitted in electronic pulses	Discrete transmission of data.
6	Amplitude & frequency vary together	Amplitude & frequency did not vary together, or always remains constant
7	Less tolerant to noise, make use of bandwidth	More tolerant to noise.

⇒ In science & engineering the mathematical models of signal are in 'deterministic' and 'stochastic'.

* Deterministic :- A model is said to be deterministic, if there is no uncertainty about its time - dependent behavior at any instant of time.

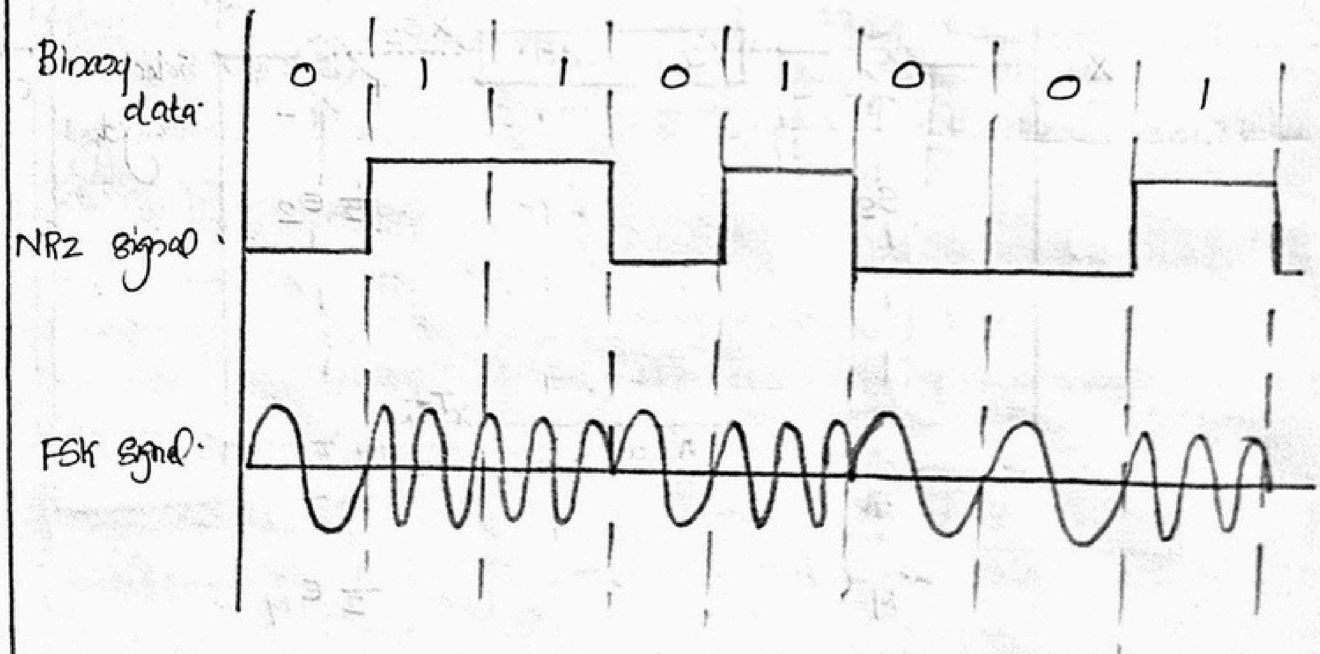
MODULE 5

Amplitude shift Keying (ASK).



The amplitude shift keying is also called On-off keying (OOK). This is simplest digital modulation technique. changes in both amplitude & phase of carrier are combined to produce amplitude phase keying.

Frequency shift keying.



... symbols.

Phase Shift Keying

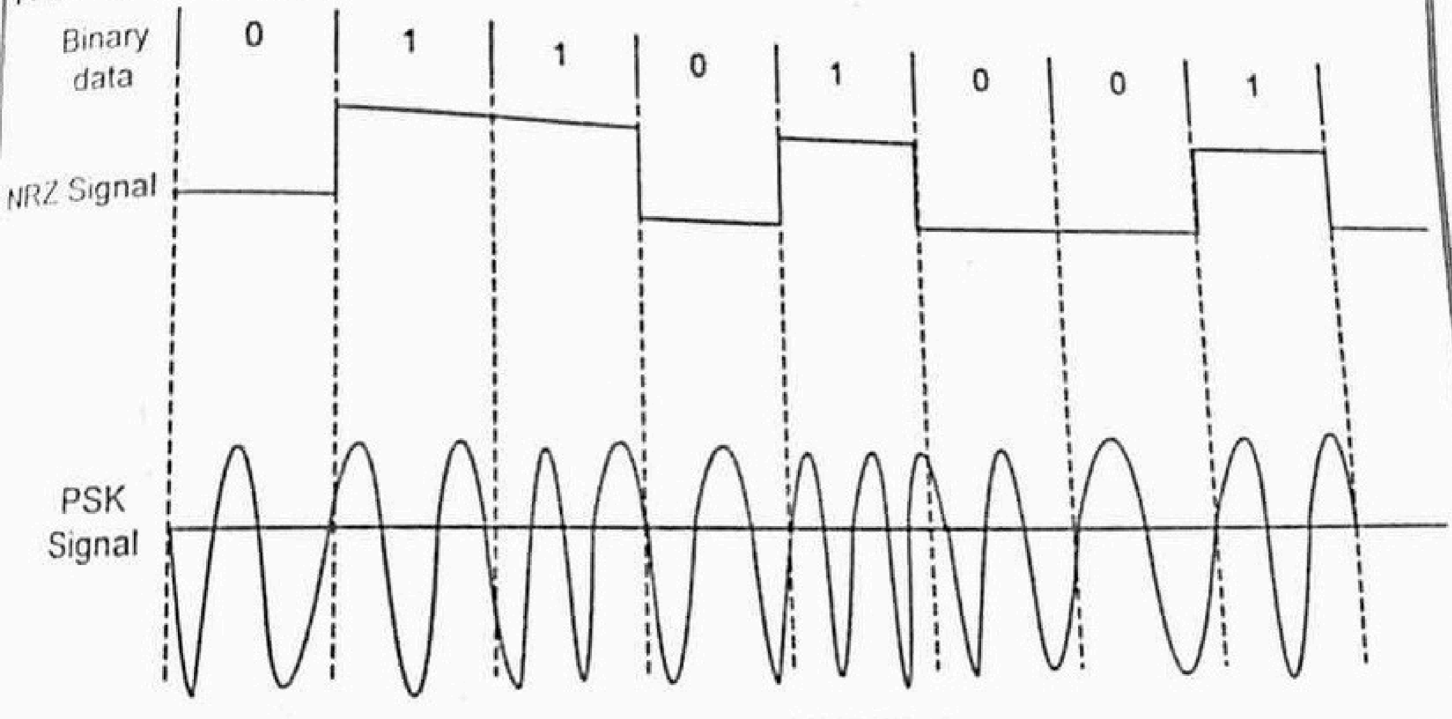


Fig. 8.3. Phase Shift Keying

The choice is made in favor of the scheme that attains as m[illegible] of the following goals as possible.

1. Maximum data rate
2. Minimum probability of symbol error.
3. Minimum transmitted power
4. Minimum channel bandwidth
5. Maximum resistance to interfering signals.
6. Minimum circuit complexity.

COMPARISON BETWEEN COHERENT & NON-COHERENT DETECTI[ON]

Sl No	Coherent detection	Non-coherent detection
1.	Receiver has exact knowledge of the carrier wave is phase reference	knowledge of the carrier wa[ve] phase is not required
2.	It is also called as synchronous detection	It is also called as envelo[pe] detection.
3.	The receiver is phase locked to the transmitter	No phase synchronization between local oscillator us[ed] in the receiver.
4.	Error probability is less	Error probability is hig[h]
5.	Receiver is more complicated	Receiver is less complica[ted]

Coherent Binary phase shift keying (BPSK)

Here the phase of the carrier signal is varied according to the binary input signal.

Carrier signal, $C(t) = A\cos(2\pi f_0 t + \phi)$

If the input bit is 1 $\rightarrow \phi = 0$.

bit is 0 $\rightarrow \phi = 180$.

$S_1(t) = A\cos(2\pi f_c t)$ for symbol 1

$S_2(t) = -A\cos(2\pi f_c t)$ for symbol 0.

Each signal is transmitted Tb second duration.

$A \rightarrow$ Amplitude of the signal.

$f_c \rightarrow$ Carrier frequency

The amplitude 'A' is represented bits per energy.

$E_b \rightarrow$ Energy per bit.

$$E_b = \int_0^{T_b} S_1^2(t)\,dt = \int_0^{T_b} A^2\cos^2 2\pi f_c t.$$

$$\left[\cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$E_b = \frac{A^2}{2}\int_0^{T_b}(1+\cos 4\pi f_c t)\,dt.$$

$$= \frac{A^2}{2}\left[\int_0^{T_b} 1.dt + \int_0^{T_b}\cos 4\pi f_c t\,dt\right]$$

$$= \frac{A^2}{2}[T_b - 0] + 0.$$

$$E_b = \frac{A^2 T_b}{2}.$$

$$A^2 = \frac{2E_b}{T_b} \qquad A = \sqrt{\frac{2E_b}{T_b}}.$$

$\therefore\ S_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos 2\pi f_c t$ for Symbol '1'

$S_2(t) = -\sqrt{\frac{2E_b}{T_b}}\cos 2\pi f_c t$ for Symbol '0'

$0 \le t \le T_b$.

By Gram - Schmidt Orthogonalization (GSOP)

$N \le M$

$N \rightarrow$ No. of basic function $= 1$

$M \rightarrow$ No. of message $= 2$.

First basic functions,

$$\phi_1(t) = \frac{S_1(t)}{\sqrt{E_b}} = \frac{\sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t}{\sqrt{E_b}}$$

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$$

$\phi_2(t) = 0 \qquad N = 1.$

So BPSK is one dimensional.

$$\boxed{\begin{array}{l}\text{orthonormal basic function} \\ \int_0^{T_b} \phi_1^2(t)\,dt = 1\end{array}}$$

$$S_{ij} = \int_0^{T_b} S_i(t)\,\phi_j(t)\,dt \qquad \begin{array}{l} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{array}$$

$M = 2, \quad N = 1$

$i = 1, 2, \quad J = 1$

$$S_{11} = \int_0^{T_b} S_1(t)\,\phi_1(t)\,dt$$

We know that $\phi_1(t) = \dfrac{S_1(t)}{\sqrt{E_b}}$

$$S_1(t) = \phi_1(t)\sqrt{E_b}$$

$$S_2(t) = -\sqrt{E_b}\ \phi_1(t)$$

$$S_{11} = \int_0^{T_b} \sqrt{E_b}\ \phi_1(t)\,\phi_1(t)\,dt$$

$$= \sqrt{E_b} \int_0^{T} \phi_1^2(t)\,dt$$

$$= \underline{\sqrt{E_b}}$$

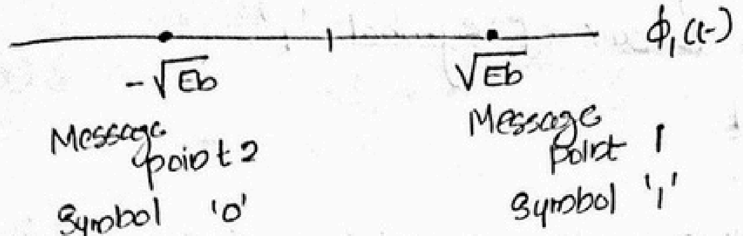
$$S_{21} = \int_0^{T_b} S_2(t)\,\phi_1(t)\,dt$$

$$= \int_0^{T_b} -\sqrt{E_b}\ \phi_1(t)\,\phi_1(t)\,dt$$

$$= \int_0^{T_b} -\sqrt{E_b}\ \phi_1^2(t)\,dt = -\sqrt{E_b}$$

The message point corresponding to $S_1(t)$ is located at $S_{11} = +\sqrt{E_b}$ and the message point corresponding to $S_2(t)$ is located at $S_{21} = -\sqrt{E_b}$.

Signal space diagram

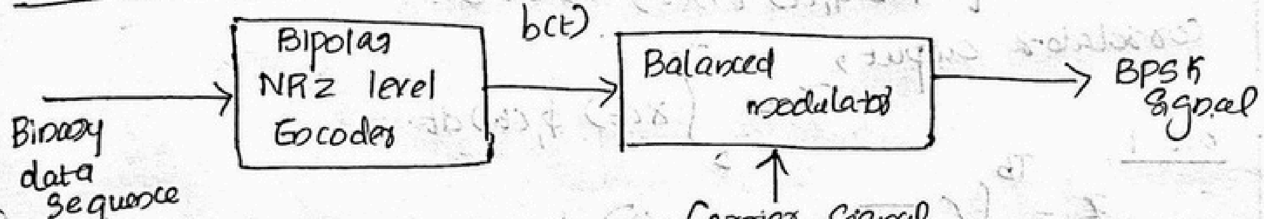


The separation between these 2 point represents isolation in symbol '1' and '0' in BPSK signal.

$$d = +\sqrt{E_b} - (-\sqrt{E_b}) = 2\sqrt{E_b}$$

- d increases the isolation between the symbols in BPSK signal is more. So probability of error reduces.

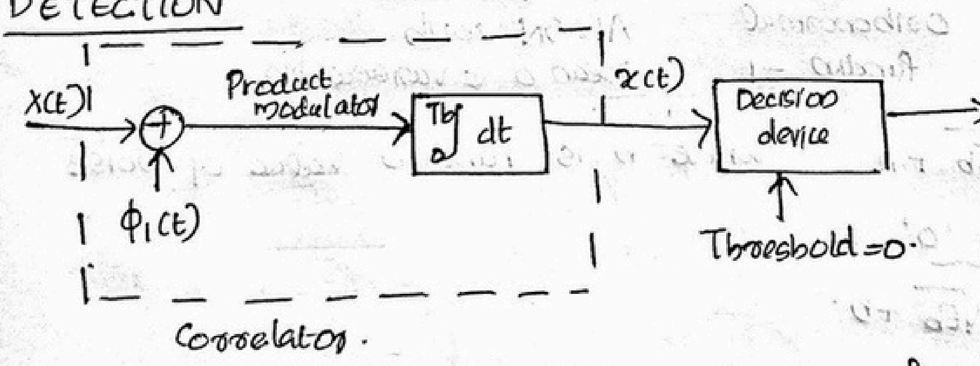
GENERATION



$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$$

- The input binary sequence sybol 1 can be represented as $+\sqrt{E_b}$ and symbol 0 represented as $-\sqrt{E_b}$. This binary wave b(t) and carrier signal are applied to balanced modulator & produce BPSK signal.

DETECTION



choose 1 if $x_1 > 0$
choose 0 if $x_2 < 0$

- To detect the original binary sequence from noisy BPSK signal x(t), the received signal from the channel is applied to a correlator
- A locally generated coherent reference signal $\phi_1(t)$, is also applied to the correlator. The output of the multiplier is integrated over one bit period (Tb). The correlator output x_1

is compared with a threshold of zero volts.

- If the correlator output is exceeded the threshold, the receiver decided in favour of symbol '1'.

$x_1 > 0 \cong 1$

→ If $x_1 < 0 \cong 0$

→ If $x_1 = 0$ the receiver make a random guess in favour of 0 or 1.

$$x(t) = \begin{cases} s_1(t) + n(t) \longrightarrow \text{bit } 1 \\ s_2(t) + n(t) \longrightarrow \text{bit } 0. \end{cases}$$

$$x(t) = \begin{cases} \sqrt{E_b}\,\phi_1(t) + n(t) \longrightarrow \text{bit } 1 \\ -\sqrt{E_b}\,\phi_1(t) + n(t) \longrightarrow \text{bit } 0. \end{cases}$$

correlator output, $x = \int_0^{T_b} x(t)\,\phi_1(t)\,dt.$

<u>bit 1</u>.

$$x = \int_0^{T_b} \left(\sqrt{E_b}\,\phi_1(t) + n(t)\right)\phi_1(t)\,dt.$$

$$= \int_0^{T_b} \sqrt{E_b}\,\phi_1^2(t)\,dt + \int_0^{T_b} n(t)\,\phi_1(t)\,dt.$$

$$= \sqrt{E_b}\cdot\underbrace{\int_0^{T_b} \phi_1^2(t)\,dt}_{\substack{\text{Orthonormal} \\ \text{function } =1}} + \underbrace{\int_0^{T_b} n(t)\,\phi_1(t)\,dt}_{\substack{\text{AWGN with} \\ \text{mean 0 \& variance } \frac{N_0}{2}}}..$$

$x = \sqrt{E_b} + n$ where n is random value of noise

For bit '0'

$x = -\sqrt{E_b} + n.$

$$\therefore x = \begin{cases} \sqrt{E_b} + n \longrightarrow \text{bit } 1 \\ -\sqrt{E_b} + n \longrightarrow \text{bit } 0 \end{cases}$$

x is a gaussian function and is characterised by its mean and variance.

Mean

$$E(x) = \begin{cases} E(\sqrt{E_b}) + E(n) \longrightarrow \text{bit } 1 \\ E(-\sqrt{E_b}) + E(n) \longrightarrow \text{bit } 0 \end{cases}$$

$$E(x) = \begin{cases} \sqrt{E_b} \longrightarrow \text{bit } 1 \\ -\sqrt{E_b} \longrightarrow \text{bit } 0 \end{cases}$$

Variance

$$E(x - \bar{x})^2 = E\left\{(\sqrt{E_b} + n - \sqrt{E_b})^2\right\}$$

$$= E[n^2] = \frac{N_0}{2} \rightarrow \text{bit } 1$$

Variance of noise.

bit 0

$$E(x - \bar{x})^2 = E\left\{(-\sqrt{E_b} + n - (-\sqrt{E_b}))^2\right\}$$

$$= E(n^2) = \frac{N_0}{2}$$

pdf of Gaussian random variable,

$$\cancel{f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}}} \quad e^{\gamma} \quad f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$X \rightarrow$ Random variable.

$\mu \rightarrow$ Mean

$\sigma^2 \rightarrow$ Variance.

when symbol '0' is transmitted,

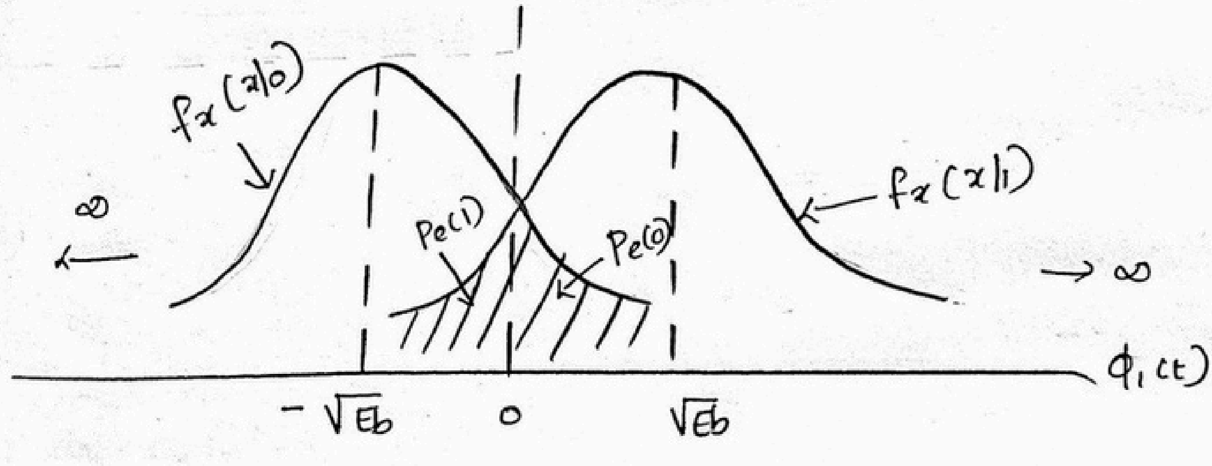
$$f_x(x/0) = \frac{1}{\sqrt{2\pi N_0/2}} e^{-\frac{(x+\sqrt{E_b})^2}{2N_0/2}}$$

$$= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x+\sqrt{E_b})^2}{N_0}}$$

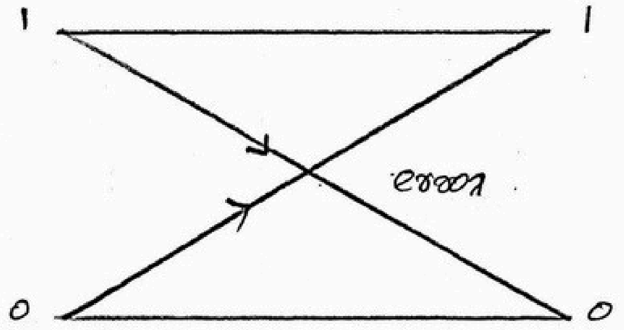
When symbol '1' is transmitted,

$$f_x(x/1) = \frac{1}{\sqrt{2\pi N_0/2}} e^{-\frac{(x-\sqrt{E_b})^2}{2N_0/2}}$$

$$f_x(x/1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x-\sqrt{E_b})^2}{N_0}}$$



→ Assume binary Symmetric channel (BSC)



Assume large length of sequence $P(0) = P(1) = \frac{1}{2}$.
Maximum likelihood detection,
average probability of error, $P_e = \frac{1}{2} P_e(0) + \frac{1}{2} P_e(1)$

$$P_e = \frac{P_e(0) + P_e(1)}{2}$$

$$P_e(0) = \int_0^{\infty} f_x(x/_0)\, dx$$

$$= \int_0^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(x+\sqrt{E_b})^2}{N_0}} dx.$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} e^{-\frac{(x+\sqrt{E_b})^2}{N_0}} dx.$$

Usually probability of error is expressed in terms of complimentary error function

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-z^2} dz$$

Put $z = \frac{1}{\sqrt{N_0}} (x + \sqrt{E_b})$ ——— (1)

Integrating x to z

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} e^{-\frac{(x+\sqrt{E_b})^2}{N_0}} dx.$$

eqn (1),
$$\sqrt{N_0}\, z = x + \sqrt{E_b}$$
$$\sqrt{N_0}\, dz = dx$$

Put $x = 0$, $z = \frac{\sqrt{E_b}}{\sqrt{N_0}}$

Put $x = \infty$, $z = \infty$.

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_{\frac{\sqrt{E_b}}{\sqrt{N_0}}}^{\infty} e^{-z^2} \sqrt{N_0}\, dz$$

$$P_e(0) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-z^2}\, dz$$

$$P_e(0) = \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} e^{-z^2}\, dz\right]$$

$$P_e(0) = \frac{1}{2}\, \mathrm{erfc}\left[\sqrt{E_b/N_0}\right]$$

Similarly, $P_e(1) = \frac{1}{2}\, \mathrm{erfc}\left[\sqrt{\frac{E_b}{N_0}}\right]$

By averaging probability of symbol error for BPSK equals

$$\boxed{P_e = \frac{1}{2}\, \mathrm{erfc}\left[\sqrt{\frac{E_b}{N_0}}\right]}$$

<u>Bandwidth</u>

$BW = (f_0 + f_b) - (f_0 - f_b)$

$BW = 2f_b$.

$f_b = \frac{1}{T_b} \Rightarrow$ Maximum frequency in baseband signal.

$f_0 =$ carrier frequency

$\therefore$ Maximum BW of PSK = Twice the highest frequency ~~contained~~ contained in baseband signal

Baud rate $= f_b$

$\Rightarrow$ <u>Coherent Binary Frequency Shift Keying (BFSK).</u>

$\rightarrow$ The frequency of the carrier is shifted according to the binary symbol. The phase of the carrier is unaffected.

$\rightarrow$ If $m = 2$.

$$S_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_i t & ; \quad 0 \le t \le T_b \\ 0 & ; \quad \text{else} \end{cases}$$

The transmitted frequency f_i

$$f_i = \frac{\omega_c + i}{T_b}$$

$\omega_c \rightarrow$ Some fixed Integer and $i = 1, 2$

Coherent Quadrature Modulation Techniques

There are 2 goals are important in the design of a digital communication system such as

1. Very low probability of error.
2. Efficient utilization of channel bandwidth.

QPSK is an example of bandwidth conserving modulation schemes for the transmission of binary data. QPSK is an extension of binary PSK.

QUADRIPHASE - SHIFT KEYING (QPSK)

In quadriphase shift keying (QPSK), the phase of the carrier takes on one of four equally spaced values, such as $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$ as shown by.

$$S_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + (2i-1)\frac{\pi}{4}\right) & , 0 \leq t \leq T \\ 0 & , \text{elsewhere} \end{cases} \quad \text{---} (1)$$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

where,

$i = 1, 2, 3, 4$

E = is the transmitted signal energy per symbol

T = is the symbol duration $= T_b \log_2(M)$; $M = 4$, $T = 2T_b$

f_c = carrier frequency equals to $\frac{n_c}{T}$ for some fixed integer n_c

Each possible value of the phase corresponds to a unique pair of bits called a dibit. For eg, we may choose the foregoing set of phase values to represent the gray encoded set of dibits 10, 00, 01 and 11.

Using trigonometric identity, we may rewrite eqn (1) in the equivalent form of

$$S_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left((2i-1)\frac{\pi}{4}\right)\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin\left((2i-1)\frac{\pi}{4}\right)\sin(2\pi f_c t); & \text{for } 0 \leq t \leq T \\ 0 & ; \text{elsewhere} \end{cases}$$

$$\text{---} (2)$$

1. There are only two ~~orthogonal~~ orthonormal basis functions, $\phi_1(t)$ and $\phi_2(t)$. The appropriate forms for $\phi_1(t)$ and $\phi_2(t)$ are

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad \text{and} \qquad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \qquad 0 \le t \le T$$

$$S_i(t) = \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right]\phi_1(t) - \sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right]\phi_2(t)$$

2. There are 4 message points and the associated signal vectors are defined by

$$S_i = \begin{bmatrix} \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right] \\ -\sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right] \end{bmatrix}$$

$N=2$
$M=4$
$i=1,2,3,4$
$N \le M$

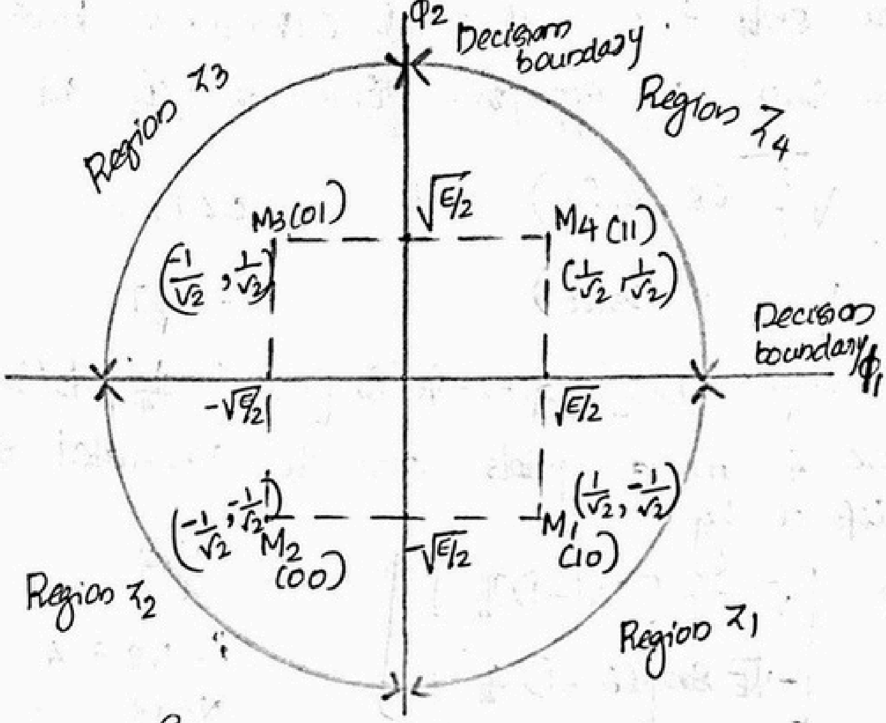
S_{i1} and S_{i2} are called elements of the signal vectors and their values are shown in below table. The first two columns of this table give the associated dibits and phase of the QPSK signal.

Signal Space characterization of QPSK			
Input dibit	Phase of QPSK signal	Coordinates of message points	
		S_{i1}	S_{i2}
10	$\pi/4$	$+\sqrt{\frac{E}{2}}$	$-\sqrt{\frac{E}{2}}$
00	$3\pi/4$	$-\sqrt{\frac{E}{2}}$	$-\sqrt{\frac{E}{2}}$
01	$5\pi/4$	$-\sqrt{\frac{E}{2}}$	$+\sqrt{\frac{E}{2}}$
11	$7\pi/4$	$+\sqrt{\frac{E}{2}}$	$+\sqrt{\frac{E}{2}}$

Geometrical Representation.

A QPSK signal is characterized by having a 2 dimensional signal constellation ($N=2$) and 4 message points ($M=4$) as shown below.

The decision regions are quadrants whose vertices coincide with the origin. These regions are marked as Z_1, Z_2, Z_3 and Z_4



Signal space diagram

The received signal $x(t)$ is defined by

$$x(t) = S_i(t) + w(t) \qquad 0 \le t \le T$$
$$i = 1, 2, 3, 4$$

where
$w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $\frac{N_0}{2}$.
The observation vector x has 2 elements x_1 and x_2 that are defined by.

$$x_1 = \int_0^T x(t)\,\phi_1(t)\,dt = \overbrace{\sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right]}^{S_1(t)} + w_1$$

and

$$x_2 = \int_0^T x(t)\,\phi_2(t)\,dt = \overbrace{-\sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right]}^{S_2(t)} + w_2.$$

where $i = 1, 2, 3, 4$

GENERATION

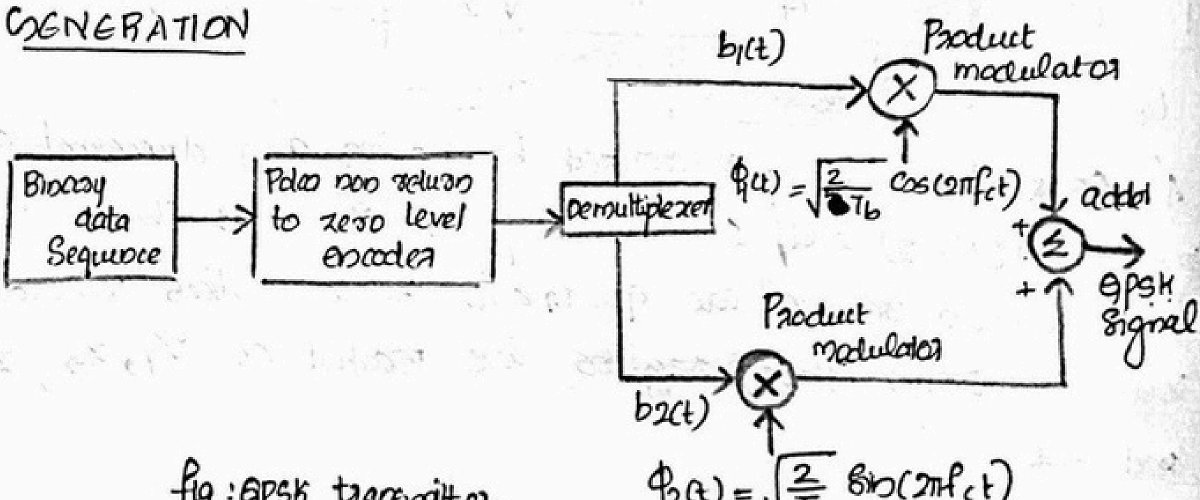


fig: QPSK transmitter.

NRZ Encoder

The input binary sequence is first transformed into polar form by a NRZ encoder. Thus symbols 1 and 0 are represented by $+\sqrt{E_b}$ and $-\sqrt{E_b}$

Demultiplexer

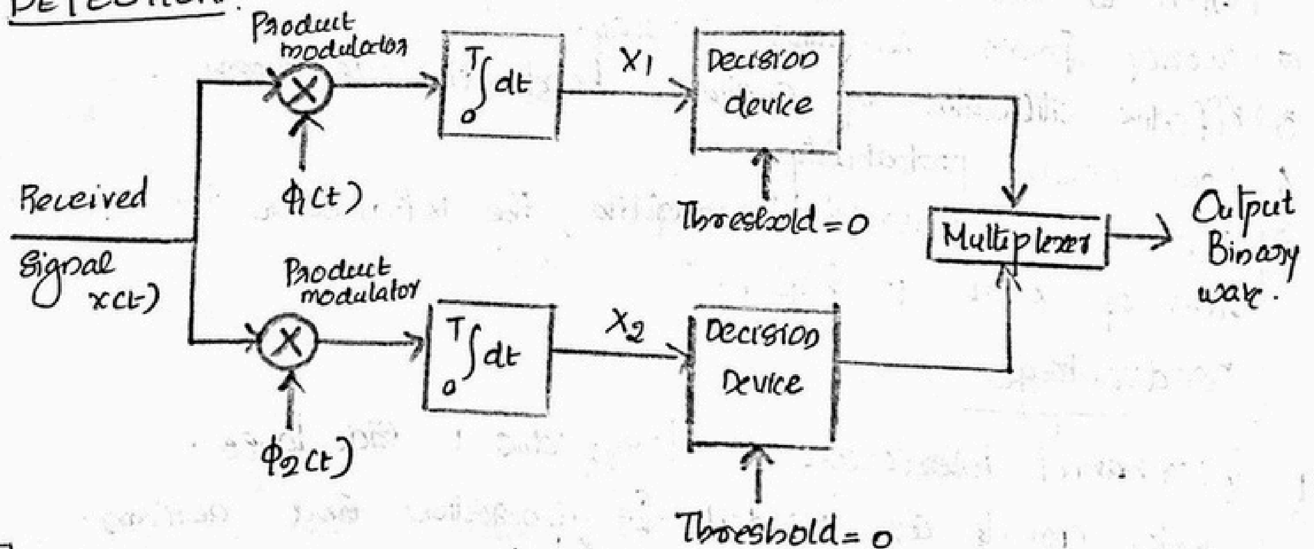
The NRZ encoder output is given to demultiplier to divide the binary wave into 2 seperate binary waves consisting of the odd and even numbered input bits. These 2 binary waves are denoted by $b_1(t)$ and $b_2(t)$

The amplitudes of $b_1(t)$ & $b_2(t)$ equal S_{i1} and S_{i2}, depending on the particular dibit that is being transmitted. The 2 binary waves $b_1(t)$ & $b_2(t)$ are used to modulate a pair of quadrature carriers or orthonormal basis functions $\phi_1(t)$ equal to $\sqrt{\frac{2}{T}}\cos(2\pi f_c t)$ and $\phi_2(t)$ equal to $\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$. The two binary waves are added to produce the desired QPSK wave.

Symbol duration

For a QPSK symbol duration is twice as long as the bit duration T_b of the input binary wave (ie) for a given rate $\frac{1}{T_b}$, a QPSK wave requires half the transmission bandwidth of the corresponding binary PSK wave.

DETECTION



The QPSK receiver consists of a pair of correlators with a common input and supplied with a locally generated pair of coherent reference signals $\phi_1(t)$ & $\phi_2(t)$.

The correlator outputs x_1 and x_2 are each compared with a threshold of zero volts.

1. If $x_1 > 0$, a decision is made in favor of symbol 1 for the upper or in-phase channel output, but if $x_1 < 0$ a decision is made in favor of symbol 0.

2. If $x_2 > 0$, a decision is made in favor of symbol 1 for the lower or quadrature channel output, but if $x_2 < 0$, a decision is made in favor of symbol 0.

Multiplexer.

The 2 binary sequence x_1 and x_2 are combined in a multiplexer to reproduce the original binary sequence at the transmitter input with the minimum probability of symbol error.

Bandwidth

In QPSK the two waveforms $\phi_1(t)$ and $\phi_2(t)$ form the baseband signals. One bit period for both these signals is equal to $2T_b$. Therefore bandwidth of QPSK is

$$BW = 2 \times \frac{1}{2T_b} \quad \text{or} \quad BW = f_b$$

The bandwidth of QPSK signal is half of the bandwidth of PSK signal.

Advantages.

1. For the same bit error rate, the bandwidth required by QPSK is reduced to half as compared to BPSK
2. carrier Power remains constant
3. Effective utilization of available bandwidth is possible.
4. Low error probability
5. Because of reduced bandwidth, the information transmission rate of QPSK is higher.

Disadvantages

1. Interchannel interference is large due to side lobes
2. Complex circuits are needed for generation and detection.

Probability of error

$$x(t) = s_i(t) + w(t) \quad ; \quad 0 \le t \le T$$
$$i = 1, 2, 3, 4$$

$w(t) \rightarrow$ AWGN with zero mean and variance $N_0/2$.

$$x_1 = \int_0^T x(t)\,\phi_1(t)\,dt$$

$$= \int_0^T (s_i(t) + w(t))\,\phi_1(t)\,dt$$

$$= \int_0^T \left(s_i(t)\phi_1(t) + w(t)\phi_1(t)\right) dt$$

$$= \int_0^T s_i(t)\,\phi_1(t)\,dt + \int_0^T w(t)\,\phi_1(t)\,dt$$

$$= \int_0^T s_i(t)\,\phi_1(t)\,dt + n$$

$$= \int_0^T \left\{\sqrt{E}\cos\left((2i-1)\frac{\pi}{4}\right)\phi_1(t) - \sqrt{E}\sin\left((2i-1)\frac{\pi}{4}\right)\phi_2(t)\right\}\phi_1(t)\,dt + n$$

$$= \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right]\underbrace{\int_0^T \phi_1^2(t)\,dt}_{1} - \sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right]\underbrace{\int_0^T \phi_1(t)\,\phi_2(t)\,dt}_{0} + n$$

$$\boxed{x_1 = \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right] + n} \quad i = 1, 2, 3, 4$$

$$x_2 = \int_0^T x(t)\,\phi_2(t)\,dt = \int_0^T (s_i(t) + w(t))\,\phi_2(t)\,dt$$

$$= \int_0^T s_i(t)\,\phi_2(t)\,dt + \int_0^T w(t)\,\phi_2(t)\,dt$$

$$= \int_0^T s_i(t)\,\phi_2(t)\,dt + n$$

$$= \int_0^T \left\{\sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right]\phi_1(t) - \sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right]\phi_2(t)\right\}\phi_2(t)\,dt + n$$

$$= \int_0^T \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right]\phi_1(t)\,\phi_2(t)\,dt + \int_0^T \sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right]\phi_2^2(t)\,dt + n$$

$$\boxed{x_2 = \sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right] + n} \quad i = 1, 2, 3, 4$$

Assume $s_4(t)$ is transmitted for i/p dibit '11'

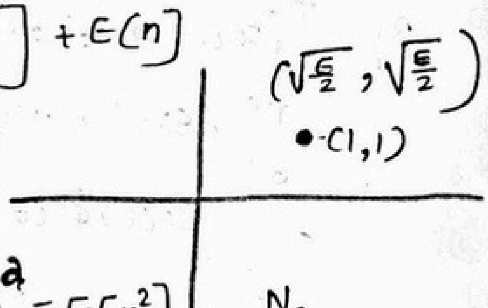
$$x_1 = \sqrt{\frac{E}{2}} + n$$
$$x_2 = \sqrt{\frac{E}{2}} + n$$

Mean $E(x_1) = E\left[\sqrt{\frac{E}{2}} + n\right]$

$$\underline{\underline{E(x_1) = \sqrt{\frac{E}{2}}}}$$

$$E(x_2) = E\left[\sqrt{\frac{E}{2}} + n\right] = E\left[\sqrt{\frac{E}{2}}\right] + E(n)$$

$$\underline{\underline{E[x_2] = \sqrt{\frac{E}{2}}}}$$



Variance.

$$Var(x_1) = E[x-\bar{x}]^2 = E\left[\sqrt{\frac{E}{2}} + n - \sqrt{\frac{E}{2}}\right]^2 = E[n^2] = \frac{N_0}{2}$$

$$Var[x_2) = E[x-\bar{x}]^2 = E\left[\sqrt{\frac{E}{2}} + n - \sqrt{\frac{E}{2}}\right]^2 = E[n^2] = \frac{N_0}{2}.$$

The probability of a correct decision P_c equal the conditional probability of the joint even $x_1 > 0$ and $x_2 > 0$. given that signal $S_4(t)$ was transmitted.

Probability of correct decision, $P_c = p(x_1 > 0)\, p(x_2 > 0)$

$$P_c = \int_0^{\infty} \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{(x_1 - \sqrt{E/2})}{N_0}} \cdot dx_1 \cdot \int_0^{\infty} \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{(x_2 - \sqrt{E/2})}{N_0}}\, dx_2 .$$

$$\left\{ \begin{array}{l} f_x(x) = \frac{1}{2\pi\sigma^2}\, e^{-(x-\mu)^2/2\sigma^2} \\ \sigma = \text{Variance} = \frac{N_0}{2} \quad , \mu = \text{mean} = \sqrt{E/2} \end{array} \right\}$$

$$P_c = P(x_1 > 0)\, P(x_2 > 0)$$

$$= \int_0^{\infty} f_x(x_1)\, dx_1 \cdot \int_0^{\infty} f_x(x_2)\, dx_2 .$$

$$z = \frac{x_1 - \sqrt{E/2}}{\sqrt{N_0}.}$$

$$z\sqrt{N_0} = x_1 - \sqrt{\frac{E}{2}}$$

$$x_1 = 0 \qquad z = \sqrt{\frac{E}{2N_0}}$$

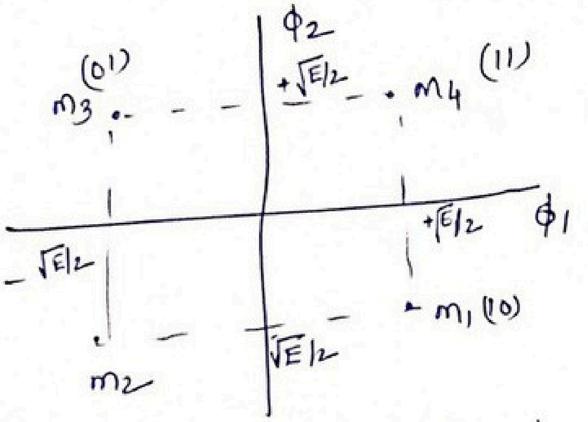
$$x_1 = \infty \qquad z = \infty$$

$$P(x_1 > 0) = \frac{1}{\sqrt{\pi N_0}} \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} e^{-z^2} \sqrt{N_0}\, dz$$

Comparison between digital modulation techniques

Sl No	Parameter	BPSK	BFSK	QPSK
1.	Variable characteristic	Phase	Frequency	phase
2.	Equation of transmitted signal s(t)	$s(t) = \sqrt{\frac{2Eb}{Tb}} \cos(2\pi f_c t)$	$s(t) = \sqrt{\frac{2Eb}{Tb}} \cos\left(2\pi f_c t + \frac{\pi t}{Tb}\right)$	$s(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + (2i-1)\frac{\pi}{4}\right)$ $i = 1, 2, 3, 4.$
3	Bit per symbol	One	One	Two
4.	No. of possible symbols	Two	Two	four
5.	Detection method	Coherent	non coherent	coherent
6	Minimum Euclidean distance	$2\sqrt{Eb}$	$\sqrt{2Eb}$	$2\sqrt{Eb}$
7	Minimum bandwidth	$2fb$	$4fb$	fb
8	Symbol duration	Tb	Tb	2Tb
9	Probability of error	$Pe = \frac{1}{2} erfc\left[\sqrt{\frac{Eb}{No}}\right]$	$Pe = \frac{1}{2} erfc\left[\sqrt{\frac{Eb}{2No}}\right]$	$Pe = erfc\left[\sqrt{\frac{Eb}{No}}\right]$
10	Symbol shaping function g(t)	$g(t) = \begin{cases} \sqrt{\frac{2Eb}{Tb}} & 0 \le t \le Tb \\ 0 & \text{elsewhere} \end{cases}$	$g(t) = \begin{cases} \sqrt{\frac{2Eb}{Tb}} \sin\left(\frac{\pi t}{Tb}\right), & 0 \le t \le Tb \\ 0 & \text{elsewhere} \end{cases}$	$g(t) = \begin{cases} \sqrt{\frac{E}{T}} & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$

Error probability of QPSK (BER of QPSK)



In QPSK, the received signal $x(t)$ is given by

$$x(t) = S_i(t) + w(t)$$

$$x_1 = \int_0^T x(t)\,\phi_1(t)\,dt = \sqrt{E}\cos\left[(2i-1)\pi/4\right] + w_1$$

$$= \pm\sqrt{\frac{E}{2}} + w_1$$

$$x_2 = \int_0^T x(t)\,\phi_2(t)\,dt = -\sqrt{E}\cos\left[(2i-1)\pi/4\right] + w_2$$

$$= \pm\sqrt{\frac{E}{2}} + w_2.$$

decision rule

Decide in favour of m_1 if x lies in Z_1

Decide in favour of m_2 if x lies in Z_2.

" of m_3 if x lies in Z_3.

" of m_4 if x lies in Z_4

An erroneous decision will be made if, due to noise, Suppose if m_4 was transmitted but received signal point falls outside Z_4.

To calculate the average probability of error, by the equation a QPSK system equivalent to 2 BPSK system.

W. know that Pe of BPSK $= \frac{1}{2}\, \text{erfc}\left[\sqrt{E_b/N_0}\right]$

QPSK bit energy $= \frac{E}{2}$.

In QPSK Pe for the Inphase channel $= \frac{1}{2}\, \text{erfc}\left[\sqrt{\frac{E}{2N_0}}\right] = p'$

Pe of Quadriphase channel $= \frac{1}{2}\, \text{erfc}\left[\sqrt{\frac{E}{2N_0}}\right] = p'$

Probability for correct decision in phase decision (PCI) $= 1 - p'$

or

probability for correct decision in Quadriphase channel. (PCQ)

So probability of correct decision in QPSK channel.

$$P_C = P_{CI} \cdot P_{CQ}$$

$$= (1-p')^2 = \left[1 - \frac{1}{2}\, \text{erfc}\left[\sqrt{\frac{E}{2N_0}}\right]\right]^2.$$

$$= 1 + \frac{1}{4}\, \text{erfc}^2\left[\sqrt{\frac{E}{2N_0}}\right] - 2\,\frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$

$$= 1 + \frac{1}{4}\, \text{erfc}^2\left[\sqrt{\frac{E}{2N_0}}\right] - \text{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$

Assume the signal to noise ratio.

$$SNR\ \frac{E}{2N_0} >>> 1$$

erfc of large value $\simeq$ small

So $P_C \simeq 1 - \text{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$

Error probability of symbol error. $\simeq 1 - P_C$.

$$P_e \simeq 1 - \left(1 - \text{erfc}\sqrt{\frac{E}{2N_0}}\right)$$

$$P_e \simeq \text{erfc}\sqrt{E/2N_0}$$

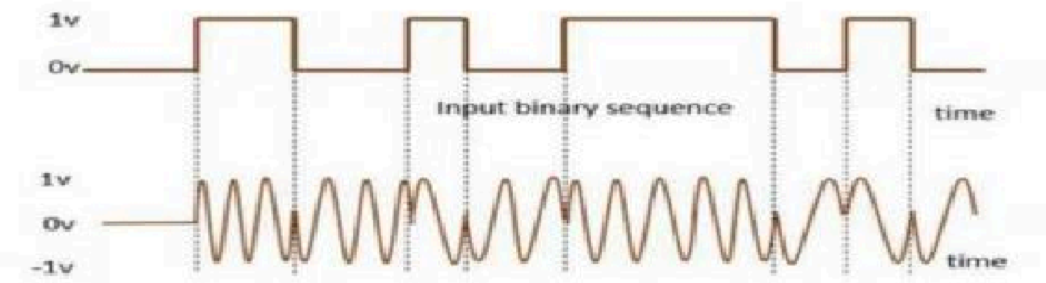
$ Bit error rate (BER) = $\frac{1}{2}$ Symbol error rate.

$\therefore$ for Bpsk

$$\boxed{BER = \frac{1}{2} \, erfc \sqrt{\frac{E_b}{N_0}}}$$

$$(BER)_{QPSK} = (BER)_{BPSK}.$$

BPSK



BPSK Modulated output wave

Figure 2-15 shows the output phase-versus-time relationship for a BPSK waveform. Logic 1 input produces an analog output signal with a 0° phase angle, and a logic 0 input produces an analog output signal with a 180° phase angle.

As the binary input shifts between a logic 1 and a logic 0 condition and vice versa, the phase of the BPSK waveform shifts between 0° and 180°, respectively.

BPSK signaling element *(t_s)* is equal to the time of one information bit *(t_b)*, which indicates that the bit rate equals the baud.

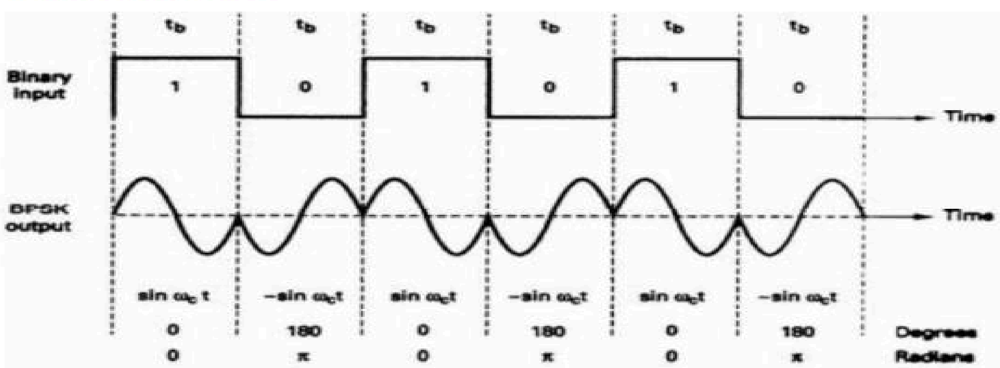
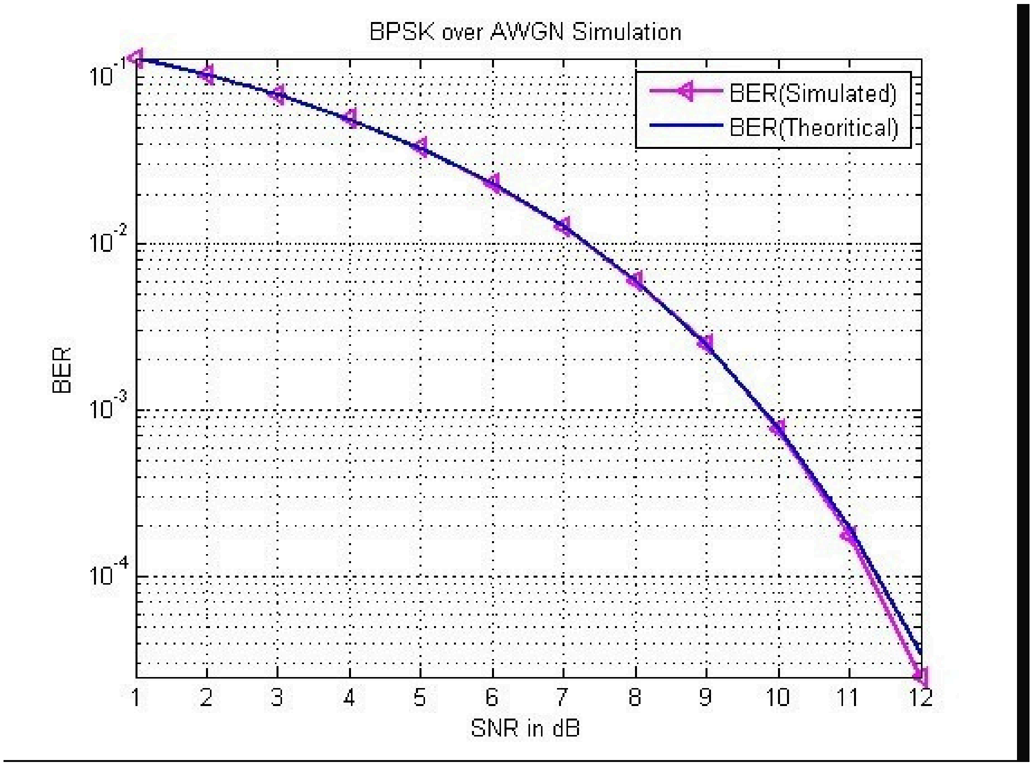
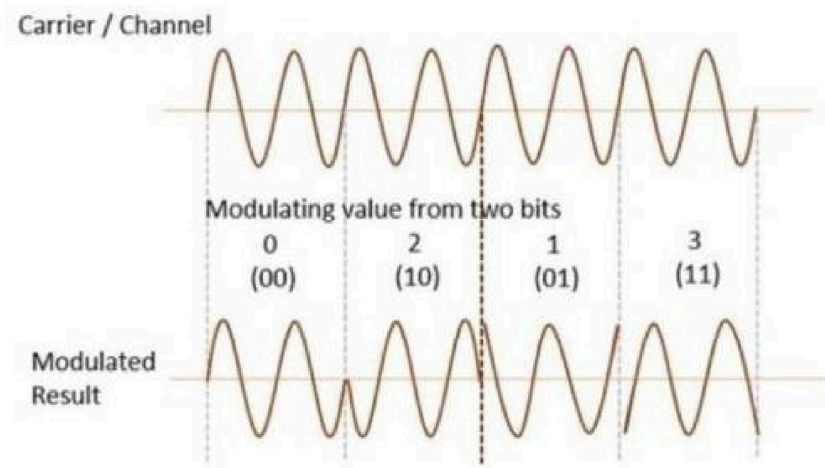


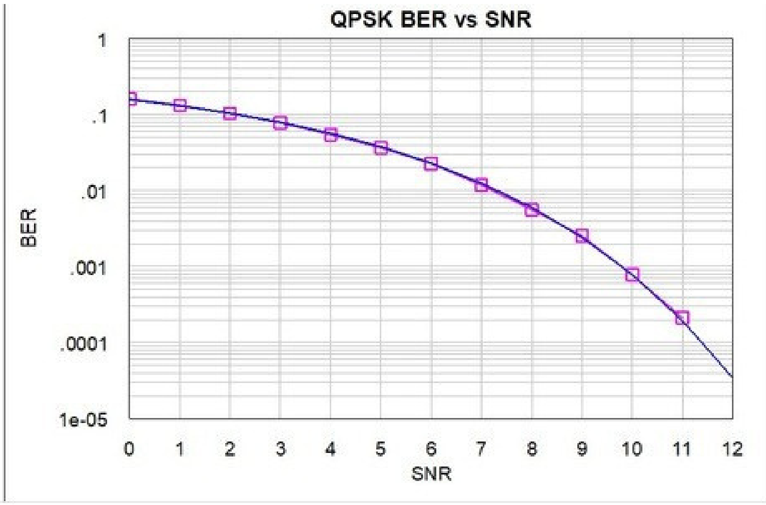
FIGURE 2-15 Output phase-versus-time relationship for a BPSK modulator



QPSK



QPSK is a variation of BPSK, and it is also a DSB-SC (Double Sideband Suppressed Carrier) modulation scheme, which sends two bits of digital information at a time, called as **bigits**.
Instead of the conversion of digital bits into a series of digital stream, it converts them into bit-pairs. This decreases the data bit rate to half, which allows space for the other users.



QPSK BER vs SNR
BER
1
.1
.01
.001
.0001
1e-05
0
1
2
3
4
5
6
7
8
9
10
11
12
SNR

this constraint is removed, and the in-phase and quadrature components are thereby permitted to be independent, we get a new modulation scheme called *M-ary quadrature amplitude modulation* (QAM). This latter modulation scheme is *hybrid* in nature in that the carrier experiences amplitude as well as phase modulation.

The passband basis functions in M-ary QAM may not be periodic for an arbitrary choice of the carrier frequency f_c with respect to the symbol rate $1/T$. Ordinarily, this aperiodicity is of no real concern. By reformulating the expression for the transmitted signal in a certain way, it is possible to eliminate the time variation of the basis functions on successive symbol transmissions, and thereby simplify implementation of the transmitter. In particular, the transmitter is made to appear "carrierless," while fully retaining the essence of the hybridized amplitude and phase modulation process. This is indeed the idea behind the carrierless amplitude/phase modulation (CAP).

Despite the differences between QAM and CAP in their implementation details, they have exactly the same signal constellations. Accordingly, they are fundamentally equivalent in performance for a prescribed receiver complexity. In what follows we first discuss QAM and then CAP.

M-ARY QUADRATURE AMPLITUDE MODULATION

In Chapters 4 and 5, we studied M-ary pulse amplitude modulation (PAM), which is one-dimensional. M-ary QAM is a two-dimensional generalization of M-ary PAM in that its formulation involves two orthogonal passband basis functions, as shown by

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \qquad 0 \le t \le T \tag{6.53}$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \qquad 0 \le t \le T \tag{6.54}$$

Let the ith message point s_i in the (ϕ_1, ϕ_2) plane be denoted by $(a_i d_{\min}/2, b_i d_{\min}/2)$, where $d_{\min}$ is the minimum distance between any two message points in the constellation, a_i and b_i are integers, and $i = 1, 2, \dots, M$. Let $(d_{\min}/2) = \sqrt{E_0}$, where E_0 is the energy of the signal with the lowest amplitude. The transmitted M-ary QAM signal for symbol k, say, is then defined by

$$s_k(t) = \sqrt{\frac{2E_0}{T}}\, a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}}\, b_k \sin(2\pi f_c t), \qquad \begin{array}{l} 0 \le t \le T \\ k = 0, \pm 1, \pm 2, \dots \end{array} \tag{6.55}$$

The signal $s_k(t)$ consists of two phase-quadrature carriers with each one being modulated by a set of discrete amplitudes, hence the name *quadrature amplitude modulation*.

Depending on the number of possible symbols M, we may distinguish two distinct QAM constellations: square constellations for which the number of bits per symbol is even, and cross constellations for which the number of bits per symbol is odd. These two cases are considered in the sequel in that order.

QAM Square Constellations

With an *even* number of bits per symbol, we may write

$$L = \sqrt{M} \tag{6.56}$$

where L is a positive integer. Under this condition, an M-ary QAM square constellation can always be viewed as the *Cartesian product* of a one-dimensional L-ary PAM constellation with itself. By definition, the Cartesian product of two sets of coordinates (representing a pair of one-dimensional constellations) is made up of the set of all possible ordered pairs of coordinates with the first coordinate in each such pair taken from the first set involved in the product and the second coordinate taken from the second set in the product.

In the case of a QAM square constellation, the ordered pairs of coordinates naturally form a square matrix, as shown by

$$\{a_i, b_i\} = \begin{bmatrix} (-L+1, L-1) & (-L+3, L-1) & \cdots & (L-1, L-1) \\ (-L+1, L-3) & (-L+3, L-3) & \cdots & (L-1, L-3) \\ \vdots & \vdots & & \vdots \\ (-L+1, -L+1) & (-L+3, -L+1) & \cdots & (L-1, -L+1) \end{bmatrix} \tag{6.57}$$

▶ EXAMPLE 6.3

Consider a 16-QAM whose signal constellation is depicted in Figure 6.17a. The encoding of the message points shown in this figure is as follows:

- Two of the four bits, namely, the left-most two bits, specify the quadrant in the (ϕ_1, ϕ_2)-plane in which a message point lies. Thus, starting from the first quadrant and proceeding counterclockwise, the four quadrants are represented by the dibits 11, 10, 00, and 01.
- The remaining two hits are used to represent one of the four possihle symbols lying within each quadrant of the (ϕ_1, ϕ_2)-plane.

Note that the encoding of the four quadrants and also the encoding of the symbols in each quadrant follow the Gray coding rule.

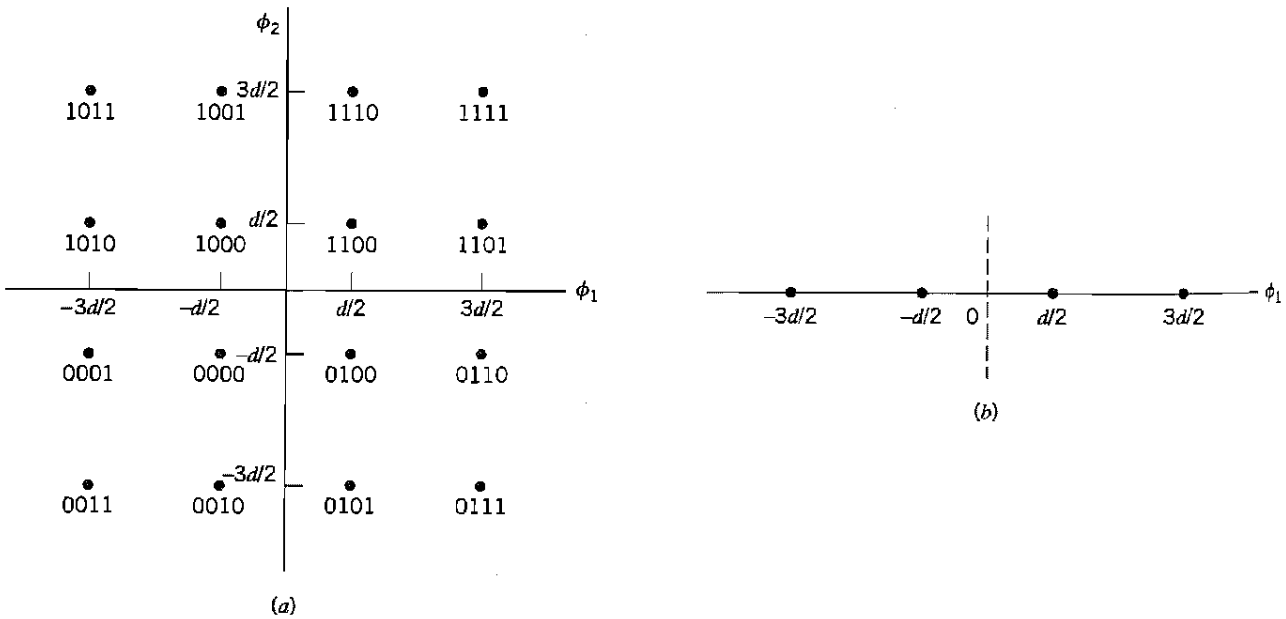


FIGURE 6.17 (a) Signal-space diagram of M-ary QAM for $M = 16$; the message points in each quadrant are identified with Gray-encoded quadbits. (b) Signal-space diagram of the corresponding 4-PAM signal.

where the multiplying factor of 2 outside the square brackets accounts for the equal contributions made by the in-phase and quadrature components. The limits of the summation and the multiplying factor of 2 inside the square brackets take account of the symmetric nature of the pertinent amplitude levels around zero. Summing the series in Equation (6.62), we get

$$\begin{aligned} E_{av} &= \frac{2(L^2 - 1)E_0}{3} \\ &= \frac{2(M - 1)E_0}{3} \end{aligned} \tag{6.63}$$

Accordingly, we may rewrite Equation (6.61) in terms of E_{av} as

$$P_e \simeq 2\left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{3E_{av}}{2(M - 1)N_0}}\right) \tag{6.64}$$

which is the desired result.

The case of $M = 4$ is of special interest. The signal constellation for this value of M is the same as that for QPSK. Indeed, putting $M = 4$ in Equation (6.64) and noting that for this special case E_{av} equals E, where E is the energy per symbol, we find that the resulting formula for the probability of symbol error becomes identical to that in Equation (6.34), and so it should.

QAM Cross Constellation

To generate an M-ary QAM signal with an odd number of bits per symbol, we require the use of a *cross constellation*. As illustrated in Figure 6.18, we may construct such a signal constellation with n bits per symbol by proceeding as follows:

- Start with a QAM square constellation with n-1 bits per symbol.
- Extend each side of the QAM square constellation by adding 2^{n-3} symbols.
- Ignore the corners in the extension.

The inner square represents 2^{n-1} symbols. The four side extensions add $4 \times 2^{n-3} = 2^{n-1}$ symbols. The total number of symbols in the cross constellation is therefore $2^{n-1} + 2^{n-1}$, which equals 2^n and therefore represents n bits per symbol as desired.

Unlike QAM square constellation, it is *not* possible to express a QAM cross constellation as the product of a PAM constellation with itself. The absence of such a factor-

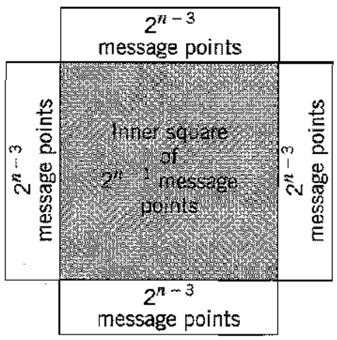


FIGURE 6.18 Illustrating how a square QAM constellation can be expanded to form a QAM cross-constellation.

ization complicates the determination of the probability of symbol error P_e incurred in the use of M-ary QAM characterized by a cross constellation. We therefore simply state the formula for P_e without proof, as shown here

$$P_e \simeq 2\left(1 - \frac{1}{\sqrt{2M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \text{ for high } E_0/N_0 \tag{6.65}$$

which agrees with the formula of Equation (6.61) for a square constellation, except for the inclusion of an extra 0.5 bit per dimension in the constellation.[5] Note also that it is not possible to perfectly Gray code a QAM cross constellation.

Carrierless Amplitude/Phase Modulation

The passband basis functions of Equations (6.53) and (6.54) assume the use of a rectangular pulse for the pulse-shaping function. For reasons that will become apparent, we redefine the transmitted M-ary QAM signal of Equation (6.55) in terms of a general pulse-shaping function $g(t)$ as

$$s_k(t) = a_k g(t - kT)\cos(2\pi f_c t) - b_k g(t - kT)\sin(2\pi f_c t), \qquad \begin{array}{l} 0 \le t \le T \\ k = 0, \pm 1, \pm 2, \ldots \end{array} \tag{6.66}$$

It is assumed that carrier frequency f_c has an arbitrary value with respect to the symbol rate $1/T$. On the basis of Equation (6.66), we may express the transmitted M-ary QAM signal $s(t)$ for an infinite succession of symbols as

$$\begin{aligned} s(t) &= \sum_{k=-\infty}^{\infty} s_k(t) \\ &= \sum_{k=-\infty}^{\infty} [a_k g(t - kT)\cos(2\pi f_c t) - b_k g(t - kT)\sin(2\pi f_c t)] \end{aligned} \tag{6.67}$$

This equation shows that for an arbitrary f_c, the passband functions $g(t - kT)\cos(2\pi f_c t)$ and $g(t - kT)\sin(2\pi f_c t)$ are *aperiodic* in that they vary from one symbol to another.

How can we eliminate the time variations of these passband basis functions from symbol to symbol? To answer this question, we find it convenient to change our formalism from real to complex notation. Specifically, we rewrite Equation (6.67) in the equivalent form

$$\begin{aligned} s(t) &= \operatorname{Re}\left\{\sum_{k=-\infty}^{\infty} (a_k + jb_k) g(t - kT)\exp(j2\pi f_c t)\right\} \\ &= \operatorname{Re}\left\{\sum_{k=-\infty}^{\infty} A_k g(t - kT)\exp(j2\pi f_c t)\right\} \end{aligned} \tag{6.68}$$

where A_k is a complex number defined by

$$A_k = a_k + jb_k \tag{6.69}$$

and $\operatorname{Re}\{\cdot\}$ denotes the real part of the complex quantity enclosed inside the braces. Clearly, Equation (6.68) is unchanged by multiplying the summand in this equation by unity expressed as the product of the complex exponential $\exp(-j2\pi f_c kT)$ and its complex conjugate $\exp(j2\pi f_c kT)$. We may thus rewrite Equation (6.68) in the new form

$$\begin{aligned} s(t) &= \operatorname{Re}\left\{\sum_{k=-\infty}^{\infty} A_k g(t-kT)\exp(j2\pi f_c t)\exp(-j2\pi f_c kT)\exp(j2\pi f_c kT)\right\} \\ &= \operatorname{Re}\left\{\sum_{k=-\infty}^{\infty} A_k \exp(j2\pi f_c kT) g(t-kT)\exp(j2\pi f_c(t-kT))\right\} \end{aligned} \tag{6.70}$$

Define

$$\tilde{A}_k = A_k \exp(j2\pi f_c kT) \tag{6.71}$$

$$g_+(t) = g(t)\exp(j2\pi f_c t) \tag{6.72}$$

The scalar $\tilde{A}_k$ is simply a *rotated* version of the complex representation of the coordinates of the kth transmitted symbol in the (ϕ_1, ϕ_2)-plane. Before presenting an interpretation of the complex-valued signal $g_+(t)$, we assume that the pulse-shaping function $g(t)$ is a low-pass signal whose highest frequency component is smaller than the carrier frequency f_c. Then following the material presented in Appendix 2, we recognize $g_+(t)$ as the *analytic signal*, or *pre-envelope*, representation of the band-pass signal $g(t)\cos(2\pi f_c t)$. To be more specific, we expand $g_+(t)$ as

$$\begin{aligned} g_+(t) &= g(t)\cos(2\pi f_c t) + jg(t)\sin(2\pi f_c t) \\ &= p(t) + j\hat{p}(t) \end{aligned} \tag{6.73}$$

where $p(t)$ and $\hat{p}(t)$ are defined by

$$p(t) = g(t)\cos(2\pi f_c t) \tag{6.74}$$

and

$$\hat{p}(t) = g(t)\sin(2\pi f_c t) \tag{6.75}$$

We may then say that the quadrature (imaginary) component $\hat{p}(t)$ of the analytic signal $g_+(t)$ is the *Hilbert transform* of the in-phase (real) component $p(t)$. Note that whereas the pulse-shaping function $g(t)$ is a baseband function, the in-phase and quadrature components of the corresponding analytic signal $g_+(t)$ are both passband functions. Henceforth, $g(t)$ is referred to as the *baseband pulse*, and $p(t)$ and $\hat{p}(t)$ are referred to as *passband in-phase* and *passband quadrature pulses*, respectively.

With the definitions of $\tilde{A}_k$ and $g_+(t)$ at hand, we are ready to finally redefine the transmitted signal of Equation (6.70) simply as

$$s(t) = \operatorname{Re}\left\{\sum_{k=-\infty}^{\infty} \tilde{A}_k g_+(t-kT)\right\} \tag{6.76}$$

Three important observations are noteworthy from this new formulation of the transmitted signal:

- The transmitted signal $s(t)$ appears to be carrierless.
- Since the formulation of $s(t)$ in Equation (6.76) is indeed equivalent to that of the M-ary QAM signal presented in Equation (6.67), the new formulation of $s(t)$ fully retains the hybridized amplitude and phase modulation characterizing the original M-ary QAM signal.
- The transmitted signal $s(t)$ represents a symbol-time-invariant realization of this hybrid modulation process.

For a prescribed carrier frequency f_c, the sequence of rotations described in Equation (6.71) is known. Hence, the receiver need only detect $\tilde{A}_k$, in which case we may compute the corresponding value of A_k by applying the reverse rotations as described here:

$$A_k = \tilde{A}_k \exp(-j2\pi f_c kT), \qquad k = 0, \pm 1, \pm 2, \ldots$$

In practice, however, the rotations are ignored because they do *not* have any bearing on operation or performance of the hybrid modulation system; application of the rotations is in fact necessary only when its equivalence to QAM is an issue of interest. Accordingly, we may ignore Equation (6.71) and redefine the transmitted signal of Equation (6.76) simply as

$$\begin{aligned} s(t) &= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} A_k g_+(t - kT)\right\} \\ &= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} (a_k + jb_k)(p(t - kT) + j\hat{p}(t - kT))\right\} \\ &= \sum_{k=-\infty}^{\infty} [a_k p(t - kT) - b_k \hat{p}(t - kT)] \end{aligned} \tag{6.77}$$

where, as mentioned previously, $\hat{p}(t)$ is the Hilbert transform of $p(t)$. For obvious reasons, the transmitted signal of Equation (6.77) is referred to as *carrierless amplitude/phase modulation* (CAP).[6]

Properties of the Passband In-phase and Quadrature Pulses

From the definitions of the passband in-phase and quadrature pulses given in Equations (6.74) and (6.75), we deduce the following properties:

Property 1

The passband in-phase pulse p(t) *and quadrature pulse* p̂(t) *are even and odd functions of time* t, *respectively, given that the baseband pulse* g(t) *is an even function of time* t.

This property follows directly from Equations (6.74) and (6.75).

Property 2

The passband pulses p(t) *and* p̂(t) *form an orthogonal set over the entire interval* $(-\infty, \infty)$ *as shown by*

$$\int_{-\infty}^{\infty} p(t)\hat{p}(t)\, dt = 0 \tag{6.78}$$

The reason for not restricting the integration interval in Equation (6.78) to a symbol period T is that the pulses $p(t)$ and $\hat{p}(t)$ are bandwidth-efficient for high-performing CAP systems. To prove Property 2, we transform Equation (6.78) into the frequency domain by using the Fourier transform to write

$$\int_{-\infty}^{\infty} p(t)\hat{p}(t)\, dt = \int_{-\infty}^{-\infty} P(f)\hat{P}^*(f)\, df \tag{6.79}$$

where $p(t) \rightleftharpoons P(f)$ and $\hat{p}(t) \rightleftharpoons \hat{P}(f)$. The asterisk in $\hat{P}^*(f)$ denotes complex conjugation. (The frequency function $\hat{P}(f)$ should *not* be viewed as the Hilbert transform of $P(f)$;

rather, following the usual terminology in Fourier analysis, $\hat{P}(f)$ is simply the Fourier transform of $\hat{p}(t)$.) The Fourier transform $\hat{P}(f)$ is related to the Fourier transform $P(f)$ by (see Appendix 2)

$$\hat{P}(f) = -j\ \mathrm{sgn}(f)P(f) \tag{6.80}$$

where $\mathrm{sgn}(f)$ is the signum function. The Fourier transforms $P(f)$ and $\hat{P}(f)$ have the same magnitude spectrum, but their phase spectra differ by $+90$ degrees for negative frequencies and -90 degrees for positive frequencies. We may therefore rewrite Equation (6.79) as

$$\begin{aligned}\int_{-\infty}^{\infty} p(t)\hat{p}(t)\,dt &= j\int_{-\infty}^{\infty} P(f)P^*(f)\ \mathrm{sgn}(f)\,df \\ &= j\int_{-\infty}^{\infty} |P(f)|^2\ \mathrm{sgn}(f)\,df\end{aligned} \tag{6.81}$$

Recognizing that the magnitude response $|P(f)|$ is an even function of frequency f and the signum function $\mathrm{sgn}(f)$ is an odd function of frequency f, the integral of Equation (6.81) is zero, proving Property 2.

Next, let the passband pulses $p(t)$ and $\hat{p}(t)$ be passed through a linear time-invariant channel of impulse response $h(t)$, yielding the following passband outputs:

$$u(t) = p(t) \star h(t) \tag{6.82}$$

and

$$\hat{u}(t) = \hat{p}(t) \star h(t) \tag{6.83}$$

where the symbol ★ denotes convolution. We may then formulate the third property.

Property 3

The passband pulses u(t) *and* û(t), *defined in Equations (6.82) and (6.83), form a Hilbert-transform pair and are therefore orthogonal over the entire interval* $(-\infty, \infty)$ *for any* h(t).

It is this important property that makes it possible for the CAP receiver to separate the transmitted real and imaginary symbols, a_k and b_k, given the channel output. To prove Property 3, we again use the Fourier transform (in a manner similar to Equations (6.79) and (6.81)) to write

$$\begin{aligned}\int_{-\infty}^{\infty} u(t)\hat{u}(t)\,dt &= \int_{-\infty}^{\infty} U(f)\hat{U}^*(f)\,df \\ &= \int_{-\infty}^{\infty} (P(f)H(f))(-j\ \mathrm{sgn}(f)P(f)H(f))^*df \\ &= \int_{-\infty}^{\infty} j|P(f)|^2|H(f)|^2\ \mathrm{sgn}(f)\,df \\ &= 0\end{aligned}$$

▶ EXAMPLE 6.4 Bandwidth-Efficient Spectral Shaping

The CAP signal of Equation (6.77) uses multilevel encoding via the complex scalar A_k for spectral efficiency. We may further improve the bandwidth efficiency of CAP by using a spectrally efficient formulation of the baseband pulse $g(t)$. For the selection of $g(t)$, we may draw upon the *raised-cosine family* of pulse-shaping functions discussed in Chapter 4.